

Statistics for Particles Having Internal Quantum State

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Abstract

A new kind of quantum statistics which interpolates between Bose and Fermi statistics is proposed beginning with the assumption that the quantum state of a many-particle system is a functional on the internal space of the particles. The quantum commutation relations for such particle creation and annihilation operators are derived, and statistical partition function and thermodynamical properties of an ideal gas of the particles are investigated. The application of this quantum statistics for the ensemble of extremal black holes are discussed.

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Although there were some attempts to propose a generalized quantum statistics (GQS) [1], yet Bose and Fermi statistics were believed to be unique for quite a long time. However, since the model by Wilczek [2] which is system of particles with an Aharonov-Bohm type of interaction in two dimensions, particles called anyons have been a subject of intense study and a number of different physical applications have been investigated, such as fractional quantum Hall effect (FQHE) [3] and high-temperature superconductivity [4]. The concept of anyons, which is based on the wave function arises a factor $e^{i\phi}$ as it exchanges two particles (exchange statistics), is essentially two-dimensional. Another way to define GQS has been formulated by Haldane [5], which is based on the rate at the number of the available states in a system of fixed size decrease as more and more particles add to it (exclusion statistics). This statistics, formulated without any reference to spatial dimensions, captures the essential features of the anyon statistics peculiar to two-dimensional systems. Recently, a variant notion of GQS bases on deformations of the bilinear Bose and Fermi commutation relations. Particles obeying a simple case of this type statistics (the so-called "infinite" statistics) are called quons [6], which obey the minimally deformed commutator

$$[a_i, a_j^\dagger]_q = \delta_{ij} \quad (0.1)$$

where $[A, B]_q \equiv AB - qBA$, and q is a c -number, $|q| \leq 1$. The equivalence of anyon statistics and quon statistics, Eq.(0.1) with $q = e^{i\phi}$, was proved [7] via the properties of the N -anyon permutation group. More recently, a new model of GQS, in which identical particles exhibit both Bose and Fermi statistics with respective probabilities p_b and p_f , is introduced by Medvedev [8].

In this letter we investigate a new exchange statistics beginning with the assumption that the quantum state of a many-particle system is a functional on the internal space of the particles. Three decades ago, it was shown by Finkelstein and Rubenstein [9] that, in nonlinear field theories, soliton statistics can be determined from the fact that the quantum state is a functional on the space of field configurations. If the soliton has no internal states, the eigenspaces of exchange operator are superselection sector: Bosons are forever bosons

and fermions are forever fermions; if the soliton does have internal state then the exchange operator may or may not change the field configuration, depending on whether or not the solitons are the same state [10]. These ideas were firstly applied to quantum gravity in a series of beautiful papers by Friedman and Sorkin [11]. Recently, Strominger applied these ideas to the problem of charged extremal black hole statistics and he argued that the charged extremal black holes will obey the infinite statistics with $q = 0$ on the condition that none of them are in the same internal state.

Basing on the developed ideas above, we assume at first that the wave function of such particles is more composition, *a functional*, according to its internal degrees of freedom, and suppose then that a *phase factor* the wave functional of the many-particle system arises as single exchange of a pair of particles is an *operator* [13], which is dependent on the intrinsic property of the particles, rather than an usual *c*-number. Furthermore, we consider that all the particles of the system are in the same state, and then the exchange operator will be independent of the pairs of particles and commute with any operator in the system simply because it dose neither change the physical configuration nor mix up internal states again. This operator is marked by \hat{q} in following.

Consider a system of N noninteracting such identical particles, represented by wave functional $\Psi(x_1, \dots, x_j, \dots, x_i, \dots, x_N)$. Single exchange any two particles we get

$$\Psi(x_1, \dots, x_j, \dots, x_i, \dots, x_N) = \hat{q}\Psi(x_1, \dots, x_i, \dots, x_j, \dots, x_N) \quad (0.2)$$

It is easy to show that the operator \hat{q} is both Hermitian and unitary [14], i.e., it satisfies

$$\hat{q} = \hat{q}^\dagger \quad \text{and} \quad \hat{q}^\dagger \hat{q} = \hat{q} \hat{q}^\dagger = \mathbf{1} \quad (0.3)$$

The eigenvalues of \hat{q} take on $+1$ or -1 only, and the eigenequation of \hat{q} is

$$\hat{q} |\pm 1, j\rangle = \pm 1 |\pm 1, j\rangle \quad (0.4)$$

where $|\pm 1, j\rangle$ compose a complete orthonormal set, with j denoting the degeneration degrees of freedom; and then the internal quantum state can be written

$$|\mathcal{I}n\rangle = \sum_i c_i^{(+)}|+1, i\rangle + \sum_j c_j^{(-)}|-1, j\rangle \quad (0.5)$$

with the normalization condition: $\sum_i |c_i^{(+)}|^2 + \sum_j |c_j^{(-)}|^2 = 1$. The exchange symmetry in Eq.(0.2) is generally neither symmetry nor anti-symmetry and we call it \hat{q} -symmetry below.

To study statistical properties of the system, let us first develop the notation of the occupation-number representation. For convenience, we introduce the following Hermitian projection operator for the system

$$\hat{Q} = \frac{1}{N!} \sum_P \hat{q}^{[P]} P \quad (0.6)$$

where P is permutation operator, the summation is over the $N!$ permutations of the N particles and $\hat{q}^{[P]} = \mathbf{1}$ or \hat{q} according as the permutation P is *even* or *odd*. It is easily proved [14] that

$$\hat{Q}^\dagger = \hat{Q} \quad \hat{Q}^2 = \hat{Q} \quad (0.7)$$

Let $\{\psi_i\}$ be a complete orthonormal set of single-particle states, the \hat{q} -symmetrized wave function can be expressed in the form

$$\Psi^{\hat{q}}(x_1, x_2, \dots, x_N) = C_N^{\hat{q}} \hat{Q} \{\psi_{i_1}(x_1) \psi_{i_2}(x_2) \cdots \psi_{i_N}(x_N)\} \quad (0.8)$$

where $C_N^{\hat{q}}$ is the *normalization constant* to be specified below. If there are n_1 particles in single-particle state ψ_1 , n_2 particles in single-particles state ψ_2 , etc., then not all of $N!$ terms $\hat{Q}\{\psi_{i_1}(x_1) \psi_{i_2}(x_2) \cdots \psi_{i_N}(x_N)\}$ are, in general, different. Take the permutations only among ψ_i (given $n_i > 1$) single-particle states as an example, we get $\frac{n_i!}{2}$ permutations are in the same form and the other $\frac{n_i!}{2}$ permutations are in another same form which is only different from the form above by a factor \hat{q} . Consequently, if the internal permutations of ψ_i s are not considered, one can equivalently contribute a factor $\frac{n_i!}{2}(\mathbf{1} + \hat{q})$. Thus the sum of all $N!$ permutations are equal to the sum of all the *different* permutations products a factor $(\prod_i n_i!)(\frac{\mathbf{1}+\hat{q}}{2})^T$, where T is the total number of single-particle state occupied by more than one particle. Hence Eq.(0.8) becomes

$$\Psi_{n_1, n_2, \dots, n_k}^{\hat{q}}(x_1, x_2, \dots, x_N) = C_N^{\hat{q}} \left(\prod_{i=1}^k n_i! \right) \left(\frac{1 + \hat{q}}{2} \right)^T \frac{1}{N!} \sum_{P_E} \hat{q}^{[P_E]} P_E \{ [\psi_1(x_1) \cdots \psi_1(x_{n_1})] \\ [\psi_2(x_{n_1+1}) \cdots \psi_2(x_{n_1+n_2})] \cdots [\psi_k \cdots \psi_k(x_N)] \} \quad (0.9)$$

where P_E are the permutations only among *different* single-particle states, $\hat{q}^{[P_E]} = \mathbf{1}$ or \hat{q} according as the permutation P_E is *even* or *odd* and all these permutation terms are different.

Now we introduce the unit operator, marked by $\mathbf{1}^{\hat{q}}$, for the system, and then $C_N^{\hat{q}}$ can be factorized as $C_N^{\hat{q}} = C_N \mathbf{1}^{\hat{q}}$. From Eqs.(0.7), (0.8) and (0.9), we obtain

$$\mathbf{1}^{\hat{q}} = (\Psi^{\hat{q}}, \Psi^{\hat{q}}) = |C_N|^2 \mathbf{1}^{\hat{q}} \frac{\prod_{i=1}^k n_i!}{N!} \left(\frac{1 + \hat{q}}{2} \right)^T \quad (0.10)$$

By using Eq.(0.2), one then obtain from this equation that

$$\mathbf{1}^{\hat{q}} = \begin{cases} \mathbf{1} & \text{for } n_1, n_2, \dots, n_i, \dots \leq 1 \\ \frac{1}{2}(\mathbf{1} + \hat{q}) & \text{otherwise (i.e., exist } n_i \geq 2) \end{cases} \quad (0.11)$$

and $C_N = \sqrt{\frac{N!}{\prod_{i=1}^k n_i!}}$. Thus

$$\Psi_{n_1, n_2, \dots, n_k}^{\hat{q}}(x_1, x_2, \dots, x_N) = \sqrt{\frac{\prod_{i=1}^k n_i!}{N!}} \mathbf{1}^{\hat{q}} \sum_{P_E} \hat{q}^{[P_E]} P_E \{ [\psi_1(x_1) \cdots \psi_1(x_{n_1})] \\ [\psi_2(x_{n_1+1}) \cdots \psi_2(x_{n_1+n_2})] \cdots [\psi_k \cdots \psi_k(x_N)] \} \quad (0.12)$$

is a \hat{q} -symmetrized orthonormal N -particle state.

In the occupation-number representation the state Eq.(0.12) is written $|n_1, n_2, \dots, n_k \rangle^{\hat{q}}$, where n_i is the number of particles in the state ψ_i and the superscript \hat{q} marks intrinsic property of the particle (for instance, when $\hat{q} = \mathbf{1}$ or $-\mathbf{1}$, the particle is bosonic or fermionic). Subject to $\sum_i n_i = N$, all of $|n_1, n_2, \dots \rangle^{\hat{q}}$ form a complete orthonormal set, i.e.,

$$\langle n'_1, n'_2, \dots | n_1, n_2, \dots \rangle^{\hat{q}} = \mathbf{1}^{\hat{q}} \delta_{n_1 n'_1} \delta_{n_2 n'_2} \cdots \quad (0.13)$$

$$\sum_{\{n_i\}, \sum_i n_i = N} |n_1, n_2, \dots \rangle^{\hat{q}} \langle n_1, n_2, \dots | = \mathbf{1}^{\hat{q}} \quad (0.14)$$

and compose the basis states of a Hilbert space $\mathcal{H}_N^{\hat{q}}$.

To simplify application of the grand-canonical ensemble formulation of statistical mechanics, we extend the development above to systems with an indefinite number of particles, whose corresponding Hilbert space is called Fock-like space here. It is useful to define the vacuum state, denoted $|0\rangle^{\hat{q}}$, which represents a state $|0, 0, \dots, 0, \dots\rangle^{\hat{q}}$ with zero particles. Hence, the Fock-like space can be indicated: $\mathcal{F}^{\hat{q}} = \bigoplus_{N=0}^{N=\infty} \mathcal{H}_N^{\hat{q}}$, where by definition: $\mathcal{H}_0^{\hat{q}} = |0\rangle^{\hat{q}}$. The closure relation in the Fock-like space may be written

$$\sum_{n_1, n_2, \dots, n_i, \dots} |n_1, n_2, \dots, n_i, \dots\rangle^{\hat{q}} \langle n_1, n_2, \dots, n_i, \dots| = \mathbf{1}^{\hat{q}} \quad (0.15)$$

We now introduce the annihilation operator \hat{a}_i and its Hermitian conjugate creation operator \hat{a}_i^\dagger in the Fock-like space. They are defined in term of their effects on the state vector, as follows:

$$\hat{a}_i |n_1, n_2, \dots, n_i, \dots\rangle^{\hat{q}} = \hat{q}^{(\sum_{l=1}^{i-1} n_l)} \sqrt{n_i} \mathbf{1}^{\hat{q}} |n_1, n_2, \dots, n_i - 1, \dots\rangle^{\hat{q}} \quad (0.16)$$

and

$$\hat{a}_i^\dagger |n_1, n_2, \dots, n_i, \dots\rangle^{\hat{q}} = \hat{q}^{(\sum_{l=1}^{i-1} n_l)} \sqrt{n_i + 1} \mathbf{1}^{\hat{q}} |n_1, n_2, \dots, n_i + 1, \dots\rangle^{\hat{q}} \quad (0.17)$$

where the associated factor $\hat{q}^{(\sum_{l=1}^{i-1} n_l)}$ comes from the exchange symmetry Eq.(0.2) among different single-particle states and the unit operator $\mathbf{1}^{\hat{q}}$ is written distinctly in order that the following calculation becomes clear. By using Eq.(0.17), it is easy to show that the Fock-like space $\mathcal{F}^{\hat{q}}$ also can be spanned by $|0\rangle^{\hat{q}}$ and all $\frac{(\hat{a}_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_2!}} \dots |0\rangle^{\hat{q}}$. We next define the number operator for particles in the state ψ_i by $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$. Then, one can easily find that

$$\hat{n}_i |n_1, n_2, \dots, n_i, \dots\rangle^{\hat{q}} = n_i \mathbf{1}^{\hat{q}} |n_1, n_2, \dots, n_i, \dots\rangle^{\hat{q}} \quad (0.18)$$

The total number operator is clearly $N(\hat{q}) = \sum_i \hat{n}_i = \sum_i \hat{a}_i^\dagger \hat{a}_i$. A careful application of Eqs.(0.17), (0.16) and (0.11) yields

$$[\hat{a}_i, \hat{a}_j^\dagger]_{\hat{q}} = \mathbf{1}^{\hat{q}} \delta_{ij} \quad (0.19)$$

$$[\hat{a}_i, \hat{a}_j]_{\hat{q}} = [\hat{a}_i^\dagger, \hat{a}_j^\dagger]_{\hat{q}} = 0 \quad (0.20)$$

The commutators are similar to the so-called \hat{q} -uon commutators introduced by Wu and Sun [15] in the point that the c -number q in Eq.(0.1) is replaced by a linear operator \hat{q} , but the real difference between them is that in Eq.(0.19) the creation and annihilation operator and their commutator resulting from the unit operator Eq.(0.11) are number-distribution-dependent. It is easily shown that the commutators are peculiar to the commutators of Boson when $\hat{q} = \mathbf{1}$ and Fermion when $\hat{q} = -\mathbf{1}$.

Following, we investigate the statistical properties of such particles. Naturally, the Hamiltonian of an ideal gas of such particles is given by $H(\hat{q}) = \sum_i \varepsilon_i \hat{a}_i^\dagger \hat{a}_i$, with ε_i s the single-particle energy levels. Now let us start with the grand canonical partition functional

$$\Xi(\hat{q}) = \text{Tr}_{\hat{q}} \exp\{-\beta[H(\hat{q}) - \mu N(\hat{q})]\} \quad (0.21)$$

where $\beta = 1/kT$ with T being temperature of the gas, μ is the chemical potential, and $\text{Tr}_{\hat{q}}(\dots) = \sum_{n_1, n_2, \dots}^{\hat{q} < n_1, n_2, \dots} |(\dots) | n_1, n_2, \dots >^{\hat{q}}$. From $e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{\hat{A}^n}{n!}$ and Eq.(0.2), the partition functional can be derived

$$\Xi(\hat{q}) = \prod_i \frac{1 - \frac{1-\hat{q}}{2} e^{2\beta(\mu - \varepsilon_i)}}{1 - e^{\beta(\mu - \varepsilon_i)}} \quad (0.22)$$

It is evident that, besides the intrinsic operator \hat{q} mentioned above, the physical properties of such particles are decided by internal quantum state which reflects the statistics of identical particles with internal degrees of freedom. For instance, when the internal state of the particle is the eigenstate of \hat{q} , $\sum_i c_i^{(+)} | +1, i >$ or $\sum_j c_j^{(-)} | -1, j >$, the particle displays boson or fermion. In general, the internal quantum state of such particles is Eq.(0.5), and then the particles show a new feature which interpolates between boson and fermion. Under such state, we obtain the grand partition function

$$\Xi(\delta) = \prod_i \frac{1 - \delta z^2 e^{-2\beta\varepsilon_i}}{1 - z e^{-\beta\varepsilon_i}} \quad (0.23)$$

where $\delta = \sum_j |c_j^{(-)}|^2$, $0 \leq \delta \leq 1$ and $z = e^{\beta\mu}$. The meaning of the parameter δ is similar to the probability p_f in Medvedev's model [8]. Obviously, when $\delta = 0$ or 1 , we obtain the partition function of ideal Boson or Fermion [16]. The occupation number in energy level ε_i is

$$n_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1} - \frac{2\delta}{e^{2\beta(\varepsilon_i - \mu)} - \delta} \quad (0.24)$$

In order for all possible n_i is non-negative, the chemical potential should be satisfy

$$\mu \leq 0 \quad \text{or} \quad \mu \geq \varepsilon_{max} - \frac{1}{2}kT \ln \delta \quad (0.25)$$

where ε_{max} is the highest energy level of the system. Thus the chemical potential of this ideal gas will emerge a gap on condition that $\varepsilon_{max} - \frac{1}{2}kT \ln \delta > 0$.

The thermodynamic properties can be derived straight-forwardly from partition function. The thermodynamic potential, $\Omega = -PV$, is given by

$$\Omega = -\beta^{-1} \ln \Xi(\delta) = -\beta^{-1} \sum_i [-\ln(1 - ze^{-\beta\varepsilon_i}) + \ln(1 - \delta z^2 e^{-2\beta\varepsilon_i})] \quad (0.26)$$

The total particle number of the system is

$$N = \sum_i \left[\frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1} - \frac{2\delta}{e^{2\beta(\varepsilon_i - \mu)} - \delta} \right] \quad (0.27)$$

The chemical potential μ as a function of temperature T and number density $\frac{N}{V}$ is defined implicitly by Eqs.(0.25) and (0.27). Since the summation is converted to integration as $\sum_{\mathbf{p}} \rightarrow \frac{V}{h^3} \int d^3p$, if one assumes such free particles $\varepsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m}$, m is the mass of the particle, then the density of state is given by $a(\varepsilon) = \frac{V}{h^3} 2\pi(2m)^{3/2} \varepsilon^{1/2}$ and $\sum_{\mathbf{p}} \rightarrow \int d\varepsilon a(\varepsilon)$. After some mathematical manipulation [16], we get

$$\frac{PV}{kT} = \frac{V}{\lambda^3} [g_{5/2}(z) - \frac{1}{2^{3/2}} g_{5/2}(z')] - \ln\left(\frac{1-z'}{1-z}\right) \quad (0.28)$$

and

$$N = \frac{V}{\lambda^3} [g_{3/2}(z) - \frac{1}{2^{1/2}} g_{3/2}(z')] + \left(\frac{z}{1-z} - \frac{2z'}{1-z'} \right) \quad (0.29)$$

where $z' = \delta z^2$, $\lambda = \frac{h}{(2\pi mkT)^{1/2}}$, and $g_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1} dx}{z^{-1} e^x - 1}$ is the Bose-Einstein integrals. In the classical Boltzmann limit $z \ll 1$ the equation of the state taked the form of the *virial expansion* to the third order is

$$PV = NkT \left\{ 1 + \frac{1}{2} \left(\frac{\delta}{\sqrt{2}} - \frac{1}{2\sqrt{2}} \right) \frac{Nh^3}{V(2\pi mkT)^{3/2}} + \left[\left(\frac{\delta}{\sqrt{2}} - \frac{1}{2\sqrt{2}} \right)^2 - \frac{2}{9\sqrt{3}} \right] \frac{N^2 h^6}{V^2 (2\pi mkT)^3} \right\} \quad (0.30)$$

There is an effective attraction between such ideal gas when $\delta < 1/2$ and effective repulsion when $\delta > 1/2$. In the case $\delta = 1/2$, a weak attraction still exists due to the third term.

As was mentioned above, in the case that none of them are in the same internal state (where the internal state-number must be larger than the particle-number), charged extremal black holes will obey the infinite statistics with the deformation parameter $q = 0$ [12]. Correspondingly, in the case that all of them are in the same internal state (with no restriction on internal state-number), the black holes will obey the new kind of statistics. Consider a collection of charged, nonrotating extremal black holes which have the same internal state. In the dilute nonrelativistic gas approximation, if gravitational and electrostatic interactions reach equilibrium via Hawking radiation, then the system is a gas of *noninteracting neutral particles*. So its thermodynamical properties are governed by the particle statistical properties, alone. According to Eq.(0.30), if the internal state of the charged extremal black holes satisfy $\delta \leq 1/2$, then they will experience *weak statistical attraction*. Hence, one may also expect that the black holes will form clusters as expected by Medvedev [8].

In summary, we have introduced a new quantum statistics, which interpolates between Bose and Fermi statistics smoothly, via the assumption that all of the particles are in the same internal quantum state reflecting the statistics of identical particles with internal degrees of freedom. A Fock-like Space of this system is obtained and the quantum commutation relations for such particle creation and annihilation operators are derived. Partition function and thermodynamical properties of an ideal gas of the particles are investigated. Finally, we discussed the implication of this statistics to the collection of charged extremal black holes having the same internal quantum state.

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